Method of Lagrange multipliers:—

How to maximize a function of many variables Subject to the constraint on the variables?

i.e. ~~the~~ functions $f(x_1, x_2, --- x_m)$ and variables are connected by constraint or equations by

$$g_1(x_1, x_2, -- x_m) = 0$$
$$g_2(x_1, x_2, --- x_m) = 0$$
$$\vdots$$

Then we use method of Lagrange undetermined multiplier to maximize these type of function. In statistical mechanics we will see such type of function where variables have constraint.

~~Let us assume that~~ If constraint are not on variable for the function f, then the maximum of $f(x_1, x_2 -- x_m)$ is given by

$$\delta f = \sum_{i=1}^{m} \left(\frac{\partial f}{\partial x_i}\right)_0 \delta x_i = 0 \quad \text{—— (1)}$$

where subscript zero in eq (1) denotes that eqn. equals zero only when partial derivatives are evaluated at extremum (max. or min.) of f.

— we denote these values of $x_i \to x_i^0$

Thus for absence of constraints $\left(\frac{\partial f}{\partial x_i}\right) = 0$, for each i

Total m equations

and we can obtain $m\, x_i^0$ from these.

Next: when constraint are present and denoted by eqn.

$g(x_1, x_2, \ldots x_m) = 0$, we have additional eqn.

$$\delta g = \sum_{i=1}^{m} \left(\frac{\partial g}{\partial x_i}\right)_0 \delta x_i = 0 \qquad \text{--- (2)}$$

Eq. (2) represents constraint that δx_i must satisfy and one of them depends on the other $(m-1)$.

In the method of Lagrange undetermined multiplier we multiply eq (2) by some parameter, say λ and add this to eq (1) and obtain

$$\boxed{\sum_{i=1}^{m} \left(\frac{\partial f}{\partial x_i} - \lambda \frac{\partial g}{\partial x_i}\right)_0 \delta x_i = 0} \qquad \text{--- (3)}$$

In eq (3) δx_i are not independent (see eq (2))

We can take δx_i in terms of other $(m-1)$ independent ones. We can take any one of m δx_i as the dependent one

Let this is denoted by δx_α

From (3) we can define λ as

$$\lambda = \left(\frac{\partial f}{\partial x_\alpha}\right)_0 \Big/ \left(\frac{\partial g}{\partial x_\alpha}\right)_0$$

Next,
$$\left(\frac{\partial f}{\partial x_i}\right)_0 - \lambda\left(\frac{\partial g}{\partial x_i}\right)_0 = 0,$$

$$i = 1, 2, \ldots \alpha-1, \alpha+1, \ldots, m.$$

Combining $(m-1)$ eqns. ~~with the~~

$$\left(\frac{\partial f}{\partial x_i}\right)_0 - \lambda\left(\frac{\partial g}{\partial x_i}\right)_0 = 0 \quad \text{for all } i$$

For more than one constraints the above method can be generalized to

$$\boxed{\frac{\partial f}{\partial x_i} - \lambda_1 \frac{\partial g_1}{\partial x_i} - \lambda_2 \frac{\partial g_2}{\partial x_i} - \cdots = 0}$$